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Self-similar quasilattices with windows having fractal boundaries

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Abstract
The point inflation rule is presented as a powerful scheme to produce a self-similar quasilattice with a non-crystallographic point symmetry. Alternatively, the conjugate point inflation rule, which is a set map acting on the internal space, has the unique fixed point, which is a compact set, W, and the quasilattice (QL) is obtained with the projection method by taking W as the window. The fixed point, W, is the attractor of the set map. If W is topologically a disc (a ball for a three-dimensional case), its boundary is usually fractal. We investigate conditions for the boundary of the window to be a von Koch curve. A practical method of constructing windows and the relevant QLs is presented. A lot of windows with fractal boundaries are determined on the basis of this method. We introduce isomerism which has priority of rank to the mutual-local-derivability as a classification scheme for self-similar QLs. It is argued that the PIR and isomerism can be the key points of classification of self-similar QLs.

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1. Introduction

An ideal quasicrystal has a long-range positional order with a non-crystallographic point symmetry, and its structure factor includes no other components than Bragg spots (for a review of crystallography of quasicrystals, see [1]). Its structure is represented as a section of a higher dimensional periodic structure. The positions of the atoms in a quasicrystal form a quasilattice (QL), which is a Delone set with quasiperiodicity. In an 'ideal model' for the quasicrystal, the QL is a subset of the *host module* which is the projection of a higher dimensional periodic lattice, i.e., the *host lattice*, onto the *physical space*. The subset is chosen with windows (or atomic surfaces), which are compact subsets in the *internal space*, the orthogonal complement of the physical space. The Bravais lattice of the host lattice only determines the Fourier module of the QL, and the windows are indispensable for a complete specification of the structure. The boundary of a window can be fractal. In what follows such a window is called simply a *fractal window* though the window itself is not fractal. A typical non-fractal window is polygonal or polyhedral depending on the dimension of the internal space. Investigation of QLs with non-fractal windows is well developed (see [1, 2] and references cited therein) but the case with fractal windows is still rudimentary [3–10]. There are several reasons for QLs with fractal windows to be more important than those with non-fractal windows from the point of view of application of QLs to real quasicrystals [10]. No general method of obtaining QLs with fractal windows has been known.

An important property of the host module of a QL is a scaling symmetry characterized by a Pisot unit, σ, in the relevant quadratic field [1, 11], and a QL often has a self-similarity. It is usual that a self-similar QL (SSQL) has a fractal window. A self-similar quasiperiodic tiling is obtained by a substitution rule for a set of tiles. The SSQL associated with it has usually a fractal window [12, 13]. However, no systematic method of obtaining substitution rules yielding quasiperiodic tilings is known. In contrast, a systematic method of obtaining SSQLs was suggested recently [10]. We call it the *point inflation rule* (PIR) because it iteratively increases the number of the lattice points to obtain a QL (a related idea to the PIR is proposed in [14, 15]); the iteration is based on a set map acting on the physical space.[1] The window of the QL produced by a PIR is determined by the conjugate set map acting on the internal space. This allows us to determine fractal windows under certain conditions. The PIR is expected to play a central role in a classification of SSQLs. The aim of the present paper is to report these results. In order to avoid complications, we confine hereafter our arguments to planer QLs whose point groups are D_n, the dihedral group of order $2n$, where the variable n runs, throughout the present paper, on the set $\{8, 10, 12\}$. The relevant internal spaces are two dimensional (2D) for these point groups.

The present paper includes a multitude of propositions, whose proofs are, however, elementary. The paper will be exceedingly expanded if we adopt the conventional form of presentation in which the results are presented as a collection of definitions, theorems and proofs, and we adopt a different style in which presentation of proofs is suppressed.

This paper is organized as follows. The n-gonal 4D lattice and relevant modules are introduced in section 2 within the scope necessary in later sections (for details, see [1, 2, 11], [17–19]). In section 3 we first define stars and then introduce a set map associated with a star. In section 4 the cut-and-projection method of constructing QLs is briefly reviewed and then the mutual-local-derivability relationship is illustrated. We see in section 5 that SSQLs are produced by PIRs. A general argument about the interplay between windows and the relevant stars is made in section 6. It is argued in section 7 that the window of an SSQL is a von Koch island. A proto-von-Koch-island is introduced in section 8 as a convenient object for access to the von Koch island. We introduce in section 9 the concept 'isomerism', which gives a classification scheme for QLs on the basis of their windows. Two ways for computer-aided drawing of a QL produced by a PIR are presented in section 10. Several planer QLs produced by PIRs are illustrated in section 11. The final section, section 12, is devoted to summary and discussion. This paper will be more readable if its earlier sections are read in parallel with section 11.

Prior to closing this section, we list acronyms which will appear henceforth but are newly defined in the present paper: SSM (star-sigma map), PS (peripheral subset), IP (induced polygon), CD (complete division), CC (complete corral), vKI (von Koch island), FA (fundamental arc) and PvKI (proto-von-Koch-island).

[1] A note added in proof. Quite recently the author became aware that preliminary investigations on the PIR are reported in [6] and [16].

2. The n-gonal 4D lattice and relevant modules

A 2D QL with the point group D_n is obtained by the cut-and-projection method from a 4D periodic lattice, L, whose 4D point group, G, is isomorphic to D_n. The 4D space, $E(\simeq \boldsymbol{R}^4)$, embedding L is a direct sum of two 2D subspaces, i.e., the physical space $E^{\parallel}(\simeq \boldsymbol{R}^2)$ and the internal space $E^{\perp}(\simeq \boldsymbol{R}^2)$. The origin of the 4D space is chosen on a lattice point of L, and is a fixed point of G. The two subspaces are invariant against the 4D point group: $G = G^{\parallel} \oplus G^{\perp}$ with $G^{\parallel} \simeq G^{\perp} \simeq D_n$. Distinctions among G, $G^{\parallel}$ and $G^{\perp}$ will be sometimes ignored. A subset of E, $E^{\parallel}$ or $E^{\perp}$ is called a *G-symmetric subset* or, simply, a *G-subset* iff it is invariant against the relevant point group.

The $\boldsymbol{Z}$-module $L^{\parallel} := \pi^{\parallel}(L)$ with $\pi^{\parallel}$ being the projector onto $E^{\parallel}$ is a dense subset of $E^{\parallel}$; it defines a translation group acting onto $E^{\parallel}$. The module, so to speak 'Bravais module', is determined apart from an arbitrary scale factor by the point symmetry. The conjugate module $L^{\perp}$ is defined similarly with respect to $E^{\perp}$ and has similar properties. There is a natural bijection, i.e., the algebraic conjugacy, between $L^{\parallel}$ and $L^{\perp}$ as well as between $G^{\parallel}$ and $G^{\perp}$; every subset X of $L^{\parallel}$ has its conjugate $X^{\perp}(\subset L^{\perp})$ and vice versa. The point group $G^{\parallel}$ (resp. $G^{\perp}$) is an automorphism of $L^{\parallel}$ (resp. $L^{\perp}$). Moreover, $L^{\parallel}$ is invariant against a scaling operation whose scaling factor is a *Pisot unit*, τ, in the quadratic algebraic field, $\boldsymbol{Z}[\tau]$ with $\tau := 1 + \sqrt{2}$, $(1 + \sqrt{5})/2$ or $2 + \sqrt{3}$ for $n = 8$, 10 or 12, respectively. More precisely, τ is the fundamental *Pisot unit* of $\boldsymbol{Z}[\tau]$, and the multiplicative group $S^{\parallel}$ generated by τ is an automorphism of $L^{\parallel}$. Then $F^{\parallel} := S^{\parallel} \times G^{\parallel}$ is an automorphism group of $L^{\parallel}$. The module $L^{\parallel}$ acts onto $E^{\parallel}$ as a translation group, and the set, $\mathcal{G}^{\parallel} := L^{\parallel}F^{\parallel}$, is an automorphism group of $L^{\parallel}$. $\mathcal{G}^{\parallel}$ is composed of similarity transformations acting onto $E^{\parallel}$. The set, $\mathcal{G}^{\parallel}_{\sigma} := L^{\parallel}(\sigma G^{\parallel})$ with $\sigma \in S^{\parallel}$, is a subset of $\mathcal{G}^{\parallel}$. Throughout the present paper the symbol τ will represent one of the above three numbers, which number in the case is context dependent. The conjugate, $S^{\perp}$, of $S^{\parallel}$ acts onto $L^{\perp}$ and is generated by the algebraic conjugate $\bar{\tau}$ of τ; the 4D operation $\tau \oplus \bar{\tau}$ is an automorphism group of L. Remember that $\bar{\tau} = -1/\tau$ for $n = 8$ and 10 but $\bar{\tau} = 1/\tau$ for $n = 12$.

The module dimension of $L^{\parallel}$ is 4 and its generators are chosen to be the first four of the n successive vertex vectors, $\boldsymbol{a}_i$, of a regular n-gon, Δ, where the variable i runs, throughout the present paper, over the cyclic group $\boldsymbol{Z}/(n\boldsymbol{Z}) = \{0, 1, \ldots, n-1\}$. A regular n-gon is sometimes identified with the set of its vertex vectors. Then $\langle\Delta\rangle = L^{\parallel}$, where $\langle * \rangle$ stands for the module generated by a set * of vectors. The vectors, $\boldsymbol{a}_i$, are not uniquely determined by this condition because $L^{\parallel}$ has $F^{\parallel}$ as its automorphism group. In particular, their scaled versions, $\tau^m \boldsymbol{a}_i$, $^{\forall}m \in \boldsymbol{Z}$, are acceptable. The conjugates of $\boldsymbol{a}_i$ are given by $\boldsymbol{a}_i^{\perp} := \lambda(-1)^i \boldsymbol{a}_i$ for $n = 8$ or 12 but by $\boldsymbol{a}_i^{\perp} := \lambda \boldsymbol{a}_{3i}$ for $n = 10$, where $\lambda \in \boldsymbol{R}^{\times}$ is an indeterminate factor; the indeterminacy is derived from arbitrariness in the length unit for $E^{\perp}$. It is important that λ does not affect any property of the QL derived from L. This circumstance is similar to the gauge symmetry in the field theory, and we can fix it arbitrarily, e.g., $\lambda = 1$, if it is necessary.

A regular n-gon, Σ, whose vertex vectors are $\boldsymbol{b}_i := \boldsymbol{a}_i + \boldsymbol{a}_{i+1}$ is an important object in discussions of later sections. It is related to Δ by $\Sigma = \omega\Delta$, where ω is a similarity operation defined as the combination of the rotation through π/n and the scaling by $2\cos(\pi/n)$. It follows that $M^{\parallel} := \langle\Sigma\rangle = \omega L^{\parallel} \subset L^{\parallel}$. The two n-gons, Δ and Σ, are G-subsets of $E^{\parallel}$ and are associated with two inequivalent mirrors of $G^{\parallel}$. It can be shown that $(L^{\parallel} : M^{\parallel}) = 2$, 5 or 1 for $n = 8$, 10 or 12, respectively. It follows that ω is an automorphism of $L^{\parallel}$ if $n = 12$, and the distinction between the two types of mirrors, Δ and Σ, is not absolute for this case [11]. The group $F^{\parallel} + \omega F^{\parallel}$ is an automorphism of $L^{\parallel}$ for $n = 12$ because $\omega^2 \in F^{\parallel}$, and we denote it newly as $F^{\parallel}$. Then $\mathcal{G}^{\parallel}$ is maximal for all n among groups which are the automorphisms of $L^{\parallel}$.

In the arguments to be made in subsequent sections, the dodecagonal case exhibits several other different behaviors from those of other two cases, and our theory will be developed with the case of $n = 8$ or $n = 10$ in mind. The modification necessary for the dodecagonal case is easily found in the individual case, and we will not present everything.

Two kinds of scaled regular n-gons are defined: $\Delta_\alpha := \alpha\Delta$ and $\Sigma_\beta := \beta\Sigma$ with $\alpha, \beta \in \boldsymbol{R}^+$; we may call α and β *size factors*.

Since $L^\parallel$ is G-symmetric, it is divided into orbits with respect to $G^\parallel$. The origin itself forms an orbit. Other orbits than the origin are called *shells*. A shell is called a *special shell* if the isotropy group of the members is nontrivial. The isotropy group of a special shell is a mirror group which is of the Δ-type or the Σ-type. A Δ-type (resp. Σ-type) shell is written as Δ_ν (resp. Σ_ν) with $^\exists\nu \in \boldsymbol{Z}^\times[\tau]$. The number of the members of a shell is n or $2n$ according to the shell is special or generic, respectively. A set of shells can be ordered by the order of their radii.

The roles of the two modules $L^\parallel$ and $L^\perp$ on which two point groups $G^\parallel$ and $G^\perp$ act are perfectly changeable. Therefore, a similar argument to the above applies to $L^\perp$. The conjugate of a special (resp. generic) shell of $L^\parallel$ is special (resp. generic) shell of $L^\perp$ and vice versa. There exist general formulae, $(\nu\Delta)^\perp = \bar{\nu}\lambda\Delta$ and $(\nu\Sigma)^\perp = \bar{\nu}\lambda\tau^{-1}\Sigma$ with $^\forall\nu \in \boldsymbol{Z}[\tau]$, and, in particular, $\Delta^\perp = \lambda\Delta$ and $\Sigma^\perp = \lambda\tau^{-1}\Sigma$. Remember, however, that the order of shells is shuffled between $L^\parallel$ and $L^\perp$.

If the 4D lattice, L, is regarded as an 'integral module', its rationalization is a module defined by $\mathcal{R} := \boldsymbol{N}^{-1}L$, which is a dense subset of E. This 'fractional module' has an infinite number of generators. The projector $\pi^\parallel$ (resp. $\pi^\perp$) is a bijection from $\mathcal{R}$ onto $\mathcal{R}^\parallel := \pi^\parallel(\mathcal{R})$ (resp. $\mathcal{R}^\perp := \pi^\perp(\mathcal{R})$), so that the bijection between $L^\parallel$ and $L^\perp$ is naturally extended to that between $\mathcal{R}^\parallel$ and $\mathcal{R}^\perp$. $\mathcal{G}^\parallel$ (resp. $\mathcal{G}^\perp$) is an automorphism group of $\mathcal{R}^\parallel$ (resp. $\mathcal{R}^\perp$) as well. Let X be a finite G-subset of $\mathcal{R}^\parallel$. Then $K^\parallel := \langle X\rangle(\subset \mathcal{R}^\parallel)$ takes the form $\alpha L^\parallel$ or $\alpha M^\parallel(\subset \alpha L^\parallel)$ with $\alpha \in \boldsymbol{Q}[\tau]$; we may say that $K^\parallel$ is a *fractional ideal* of $L^\perp$. If, in particular, $\alpha \in \boldsymbol{Z}[\tau]$, $K^\parallel \subset L^\parallel$ is an integral ideal of $L^\perp$. Then $(L^\parallel : K^\parallel) = |\alpha\bar{\alpha}|^2$ or $|\alpha\bar{\alpha}|^2(L^\parallel : M^\parallel)$, respectively. In what follows, we set $\boldsymbol{x}^\parallel := \pi^\parallel(\boldsymbol{x})$ and $\boldsymbol{x}^\perp := \pi^\perp(\boldsymbol{x}),^\forall \boldsymbol{x} \in \mathcal{R}$.

If $\alpha \in \boldsymbol{Q}^\times[\tau]$ is regarded to be a scaling operation on $E^\parallel$, $\alpha^\perp := \bar{\alpha}$ is its counterpart on $E^\perp$. More generally, a map closed in $\mathcal{R}^\parallel$ has its counterpart closed in $\mathcal{R}^\perp$. If $\sigma \in S^\parallel$ is fixed, the *hat map* $\hat{X} := \sigma X$ (resp. the *tilde map* $\tilde{X} := \bar{\sigma}X$) is a simple scaling transformation acting onto $X \in E^\parallel$ (resp. $E^\perp$). It is commutable with $G^\parallel$ (resp. $G^\perp$). Introduction of the two maps is convenient to simplify various equations to appear in later sections. They belong to $\mathcal{G}^\parallel$ and $\mathcal{G}^\perp$, respectively. Note that $\tilde{X} = \sigma^{-1}X$ if $X = -X$ as is the G-symmetric case.

If two subsets, X and Y, of $\mathcal{R}^\parallel$ (or $\mathcal{R}^\perp$) are given, the vector sum of the two sets is defined by $X + Y := \{\boldsymbol{x} + \boldsymbol{y} :^\forall \boldsymbol{x} \in X,^\forall \boldsymbol{y} \in Y\}$ or, equivalently,

$$X + Y = \bigcup_{\boldsymbol{x}\in X}(\boldsymbol{x} + Y) = \bigcup_{\boldsymbol{y}\in Y}(\boldsymbol{y} + X). \qquad (1)$$

This summation satisfies commutativity and associativity. It is commutable with the automorphism, $\mathcal{G}^\parallel$ (or $\mathcal{G}^\perp$). Moreover, it is compatible with the conjugacy. Equation (1) has a simple geometrical interpretation: $X + Y$ is obtained from X by putting Y on every member of X provided that X is discrete. If X and Y are shells, the outermost shell of the set $X + Y$ is denoted as $X \vee Y$; the operation $\vee$ is not compatible with the conjugacy.

If two subsets of $E^\parallel$ (resp. $E^\perp$) are translationally equivalent with respect to the translation group $L^\parallel$ (resp. $L^\perp$), one of them is called a *copy* of the other. The cyclic group C_n is a subgroup of D_n; $D_n = C_n + \sigma_{\mathrm{m}}C_n$ with σ_{m} being a mirror. The objects, G, $G^\parallel$, etc., defined with respect to D_n have their counterparts, H, $H^\parallel$, etc., defined with respect to C_n. An element of the automorphism, $S^\perp \times H^\perp$, is written as τ^pR^i, where R is the rotation through the angle $2\pi/n$. It

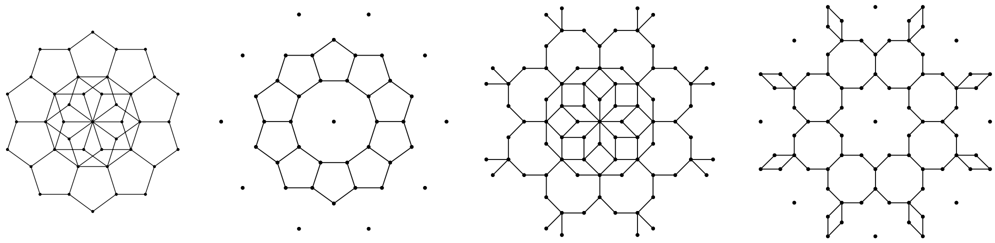

Figure 1. The left two show a conjugate pair of decagonal stars. The innermost shell of the first star is transformed to the outermost shell of the second one. The points of the outermost shell as well as the central point are isolated in the second star. The right two show a similar pair of stars but for the octagonal case.

is specified by two integers, p, the *power*, and i, the *angle number*. An equivalence relationship with respect to $H^{\parallel}$ (resp. $H^{\perp}$) distinguishes the difference in chirality in contrast to the case of $G^{\parallel}$ (resp. $G^{\perp}$).

3. Stars and relevant set maps

A *star* $\mathcal{S}$ is a finite G-subset of $L_{\parallel}$. It is called *irreducible* or *reducible* if $M^{\parallel} := \langle\mathcal{S}\rangle(\subset L^{\parallel})$ is identical to $L_{\parallel}$ or not, respectively; $\mathcal{S}$ is always irreducible with respect to $M^{\parallel}$. It is *centered* if $0 \in \mathcal{S}$. It is composed of a finite number of shells if it is not centered, and $\sharp(\mathcal{S})$, the cardinal number of $\mathcal{S}$, is a multiple of n. A Δ-type shell, Δ_{α}, and a Σ-type shell, Σ_{β}, are specified by their size factors, α and $\beta \in \boldsymbol{Z}[\tau]$. It can be shown that a generic shell is written as $\Delta_{\alpha} \vee \Sigma_{\beta}$, which is abbreviated as $\alpha \vee \beta$. Then a star is specified by the following set of the size factors of its shells: $F_{\mathcal{S}} = \{\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; \alpha'_1 \vee \beta'_1, \alpha'_2 \vee \beta'_2, \ldots, \alpha'_r \vee \beta'_r\}$ with p, q and r being the number of the relevant types of shells.

A star is a set of vectors but to graphically show it with arrows is not convenient because different vectors often overlap. It is more convenient to show it as a network whose nodes form $\mathcal{S}$ as a point set; the condition for two nodes in $\mathcal{S}$ to be bonded is that their separation is one of the generators, $\boldsymbol{a}_i$s. The bonds will cross heavily if the length of the generators is comparable to the size of $\mathcal{S}$ but all the sites will be isolated if the length is too long or too short. We can exploit the redundancy in the length of the generators to obtain a reasonable network. We will sometimes use 'motif' as a synonymy of 'star' and 'network'. For example, two decagonal motifs presented in figure 1 are specified in that order by $F_{\mathcal{S}} = \{\tau^{-1}, 1, \tau; 1\}$ and $F_{\mathcal{S}} = \{1, \tau, \tau^2; 1\}$, respectively.

If two stars are given, their sum is a star as well. The sum can be interpreted geometrically as an operation for the relevant two motifs; it is obtained from one of them by centering another motif onto every node of the other. Motifs put on different nodes can interpenetrate into each other on this procedure. Two interpenetrating motifs match well because they are subsets of $L^{\parallel}$.

A star can be defined with respect to the internal space as well. A star $\mathcal{S}$ in the physical space has its internal space counterpart, $\mathcal{S}^{\perp}$, i.e., the conjugate of $\mathcal{S}$ (see figure 1). It is a general trend that an inner shell of $\mathcal{S}$ is transformed to an outer shell of the conjugate star, $\mathcal{S}^{\perp}$, and vice versa. It is usual that a reasonably connected network on $\mathcal{S}$ is mapped to that with heavy crossings because it is folded on the mapping. Then it is more convenient to choose different generators to form a network on $\mathcal{S}^{\perp}$. It is not rare that $\mathcal{S}$ and its conjugate, $\mathcal{S}^{\perp}$ are the same, provided that, if necessary, the 'gauge', λ, is chosen appropriately. Then we may

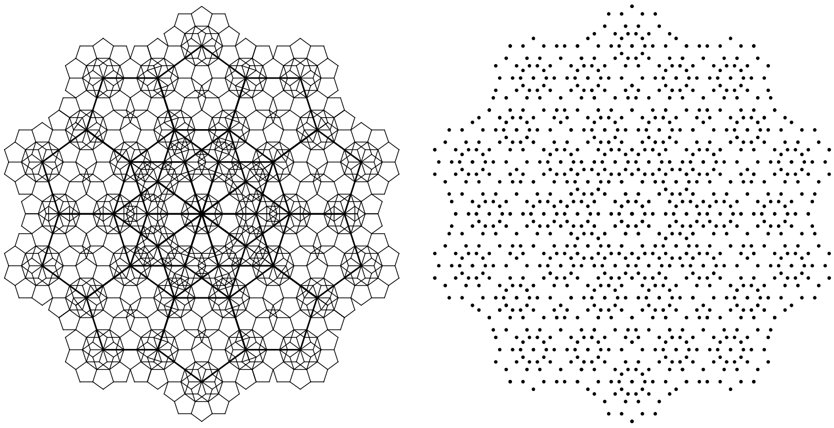

Figure 2. Left: the thick lines show the same motif, $\mathcal{S}$, as the leftmost of figure 1 and thin lines the next generation, $\mathcal{S}_2$, where $\sigma = \tau^3 (= 2 + \sqrt{5} \approx 4.236)$. Right: the nodes of the second generation motif, $\mathcal{S}_2$, on the left.

say that $\mathcal{S}$ is self-conjugate. For example, a star derived from the leftmost one in figure 1 by deleting the innermost shell is self-conjugate.

If the point group, D_n, is specified, we can choose a star $\mathcal{S}$ and a member $\sigma (= \tau^p)$ of $S^{\|}$, where p is a non-negative integer. The *star-sigma map* (SSM) defined for the pair $\{\mathcal{S}, \sigma\}$ is a set map acting on $X \subset E^{\|}$:

$$\Phi(X) := \mathcal{S} + \hat{X}. \tag{2}$$

This map plays a central role in the present paper. It is commutable with $\mathcal{G}^{\|}$. Each of the three modules, $K^{\|} \subset \mathcal{R}^{\|} \subset E^{\|} (\simeq \boldsymbol{R}^2)$ with $K^{\|}$ being a fractional ideal of $L^{\|}$, is invariant against the map. If $\mathcal{S} = \{0\}$, the SSM is reduced to the hat map. On the other hand, if $\sigma = 1$, then $\Phi(X) = \mathcal{S} + X$, which we may call the *S-map*. If the S-map follows the hat map, the resulting composition map is an SSM. The hat map, the S-map and the SSM have simple geometrical interpretations. Note that the map $X \mapsto \boldsymbol{x} + \hat{X}$ included implicitly in (2) belongs to $\mathcal{G}_{\sigma}^{\|}$. Note also that a set of SSMs generate a semi-group of SSMs.

The S-map is invertible by

Lemma. *Let $\mathcal{S}$, X and Y be vector sets and assume that $\mathcal{S}$ is finite. Then the second of the following two equations follows from the first:*

$$X = \bigcup_{\boldsymbol{x} \in \mathcal{S}} (\boldsymbol{x} + Y), \qquad Y = \bigcap_{\boldsymbol{x} \in \mathcal{S}} (-\boldsymbol{x} + X). \tag{3}$$

This lemma is easily proven. Note that the first does not necessarily follow the second. Since the hat map is invertible as well, so is the SSM. The map $X \mapsto Y$ defined by the second equation above is not an SSM, and the inverse of an SSM is never an SSM unless it is the hat map.

We assume hereafter that $\sigma > 1$ unless stated otherwise. Then the map $X \mapsto \boldsymbol{x} + \hat{X}$ is an expanding similarity transformation, and we call $\Phi(X)$ the *inflation* of X. The set $M^{\|} (= \langle \mathcal{S} \rangle)$ is an invariant submodule of $L^{\|}$ with respect to Φ.

If $\mathcal{S}$ is centered, the set map Φ is monotonic in the sense that $\Phi(X) \subset \Phi(Y)$ if $X \subset Y$. If a finite set $X \subset L^{\|}$ is chosen as the initial set, an increasing series of finite subsets of $L^{\|}$ is produced by an iteration of Φ: $X_0 (:= X), X_1, X_2, \ldots$, with $X_{j+1} := \Phi(X_j)$. If we set $X_0 := \{0\}$, we obtain a series of stars, $\{\mathcal{S}_j : j \geqslant 0\}$ with $\mathcal{S}_1 = \mathcal{S}$. As an illustration, the leftmost motif (star) of figure 1 is superposed in figure 2(left) with the next generation motif,

$\mathcal{S}_2$. The nodes of $\mathcal{S}_2$ are shown in figure 2(right). We may represent the dependence of the set map (2) on $\{\mathcal{S}, \sigma\}$ explicitly as $\Phi[\mathcal{S}, \sigma]$. A double operation of the map is also a similar set map: $\Phi^2[\mathcal{S}, \sigma] = \Phi[\mathcal{S}_2, \sigma^2]$.

The *conjugate* set map to (2) acts onto $X \subset L^\perp$:

$$\Phi^\perp(X) := \mathcal{S}^\perp + \tilde{X}. \tag{4}$$

The object X on which $\Phi^\perp$ acts can be extended to a subset of $E^\perp$. The image $\Phi^\perp(X)$ is compact if so is the set $X \in E^\perp$. If $\mathcal{S}$ is centered, $\Phi^\perp$ is monotonic. The present map satisfies the following compatibility relationship with Φ:

$$[\Phi(X)]^\perp = \Phi^\perp(X^\perp), \qquad {}^\forall X \subset \mathcal{R}^\parallel. \tag{5}$$

The map $X \mapsto \boldsymbol{x} + \tilde{X}$, $X \in E^\perp$, belongs to $\mathcal{G}^\perp$. Since it is a shrinking affine transformation, the set map (4) is a *strictly self-similar scheme* (SSSS) [21]. It is known that such a scheme always has a unique fixed point, which is a compact set and the sole attractor of the map.

4. The cut-and-projection method and the MLD relationship

We assume, for simplicity, that the number of windows used in the cut-and-projection method is a one per one unit cell of L. The window W of a QL is a compact subset of $E^\perp$ such that it has a non-vanishing interior and ∂W has a null measure. It is called regular if $\partial W \cap L^\perp = \emptyset$ but singular, otherwise. We consider hereafter only the regular case. Then a QL having L as the host lattice and W as its window is the point set:

$$Q(W) := \{\boldsymbol{x}^\parallel : \boldsymbol{x} \in L \,\&\, \boldsymbol{x}^\perp \in W\}, \tag{6}$$

which is a Delone subset of $L^\parallel$, the host module. This equation is equivalent to

$$Q^\perp(W) = L^\perp \cap W, \tag{7}$$

with $Q^\perp(W)$ being the conjugate set of $Q(W)$. Since $L^\perp$ is dense, we obtain

$$W = \text{Closure}\,(Q^\perp). \tag{8}$$

It can be easily shown that

$$\boldsymbol{x}^\parallel + Q(W) = Q(\boldsymbol{x}^\perp + W), \qquad {}^\forall \boldsymbol{x} \in L, \tag{9}$$

$$\hat{Q}(W) = Q(\tilde{W}), \qquad {}^\forall \sigma \in S^\parallel. \tag{10}$$

The symbol $Q(*)$ is regarded as a set map which transforms a subset $*$ of $E^\perp$ to a subset of $L^\parallel$. This map keeps the set relationships, $\subset, \cup$, and $\cap$. Then it satisfies the following two compatibility relationships:

$$\mathcal{G}^\parallel Q = Q\mathcal{G}^\perp, \qquad \Phi Q = Q\Phi^\perp. \tag{11}$$

The second relationship implies that $Q' := \Phi(Q)$ with $Q = Q(W)$ is another QL: $Q' = Q(W')$ with $W' := \Phi^\perp(W)$. In particular,

$$\mathcal{S} + Q(W) = Q(W'), \qquad W' := \mathcal{S}^\perp + W \tag{12}$$

with $\mathcal{S}$ being a star.

QLs with a common host module are grouped into *locally isomorphic* classes (LI classes). Two QLs are LI iff their windows are translationally equivalent. Let V be a window centered on the origin. Then $Q(V)$ is taken to be the representative of the relevant LI-class provided that V is regular. We consider hereafter the case where V is regular and G-symmetric unless stated otherwise. Then $Q(V)$ has G as its global point symmetry, and every QL in the LI class has G as its macroscopic point symmetry. From now on we will deal basically with QLs whose windows are centered on the origin, and the symbol W will be used to represent such a window. We call a window *proper* if it is homeomorphic to a disc but *improper* otherwise; a topological disc will be called simply as a *t-disc*, whose boundary is a *t-circle*, i.e., a Jordan curve. We call a QL *proper* iff its window is proper. We consider throughout the present paper proper SSQLs.

A geometrical interpretation of (12) represents a local rule for the derivation, $Q(W) \mapsto Q(W')$. More generally, if two QLs on a common host module, $L^{\|}$, are presented, one of them is *locally derivable* from the other if there exists a local rule producing lattice points of the former from the latter (for local derivability, see [20]). Let $Q := Q(W)$ and $Q' := Q(W')$ be two QLs and assume that Q is locally derivable from Q'. Then there exists a star $\mathcal{S}^{\perp} \subset L^{\perp}$ such that W is obtained as the closure of the result of a finite number of operations of the union, the intersection, and the set compliment operation on the members of $\{\boldsymbol{x} + W' : \boldsymbol{x} \in \mathcal{S}^{\perp}\}$; we may write $W = g(W')$ with g being a set map characterized by $\mathcal{S}^{\perp}$. The local rule by which Q is derived from Q' is represented as an isomorphous succession of the set operations on $\{\boldsymbol{x} + Q' : \boldsymbol{x} \in \mathcal{S}\}$. However, the closure operation is unnecessary because Q is discrete. It is important in a later argument that ∂W is covered by copies of $\partial W'$:

$$\partial W \subset \bigcup_{\boldsymbol{x} \in \mathcal{S}^{\perp}} (\boldsymbol{x} + \partial W'). \tag{13}$$

A member $\boldsymbol{x}$ in $\mathcal{S}^{\perp}$ is irrelevant in this covering relationship if $\partial W \cap (\boldsymbol{x} + \partial W') = \emptyset$.

Two QLs, Q and Q', are *mutually local derivable* (MLD) if Q is locally derivable from Q' and vice versa. This is the case if, for example, Q is the image of Q' due to an S-map, i.e., $Q = \mathcal{S} + Q'$ with $\mathcal{S}$ being a star because the S-map is invertible. The local rule of the reverse derivation, $Q \mapsto Q'$ is given by

$$Q' = \bigcap_{\boldsymbol{x} \in \mathcal{S}} (-\boldsymbol{x} + Q). \tag{14}$$

That is, a site $\boldsymbol{y}$ of Q belongs to Q' iff it has neighbors in Q at all the sites which are relatively specified by $\mathcal{S}$ to $\boldsymbol{y}$. More generally, two QLs related by an SSM as $Q = \Phi(Q')$ are MLD. The MLD concept is an important classification scheme for QLs [20]. Note, however, that it is not compatible with the properness of windows; a QL with a proper window can be MLD with another with an improper window. Since there exist an infinite variety of choices for Φ in the example above, every MLD class includes an infinite number of LI-classes.

A QL is a point set. There exists sometimes a tiling such that its vertices form the QL. We may call it the *companion tiling* of the QL. It is self-similar but the relevant substitution rule may be context dependent. Even if a QL does not have its companion tiling, there exists another QL which is MLD with the original QL and has its companion tiling. This is because the QL has *finite local complexity*.

5. Self-similar QLs

We take a QL, $Q = Q(W)$, with W being the window. It is *self-similar* iff its scaled version $\hat{Q} := \sigma Q, \sigma > 1$, is its subset and, besides, Q and $\hat{Q}$ are MLD. Then $\sigma \in S^{\|}$. The local rule of the derivation, $\hat{Q} \mapsto Q$ (resp. $Q \mapsto \hat{Q}$), is called the *inflation rule* (resp. the *deflation rule*). The self-similarity of the QL is represented mathematically by a semi-group generated by σ; the QL is self-similar in a ratio which is an arbitrary power of σ. If $\sigma = \tau^p$, the QL may happen to be self-similar in the ratio $\sigma' = \tau^q$ with q $(1 \leqslant q < p)$ being a divisor of p. Therefore, we assume that the smallest σ is employed for each QL. It is important that the self-similarity concept is compatible with the MLD relationship.

We consider henceforth an SSQL, $Q = Q(W)$, with the self-similarity ratio σ. Then $\hat{Q}(W) \subset Q(W)$, and $\tilde{W} \subset W$ by (10). This is natural because $\sigma > 1$. We call henceforth $\tilde{W}$ a *miniature window*. Since $Q(W)$ and $Q(\tilde{W})$ are MLD, there exists a star, $\mathcal{S}^\perp$, and the relevant set map, g, such that $W = g(\tilde{W})$, which is rewritten with another set map $f := g\sigma^\perp$ as $W = f(W)$. We can assume that $\mathcal{S}^\perp$ is centered because $\tilde{W} \subset W$. We call the conjugate $\mathcal{S}$ of $\mathcal{S}^\perp$ a *companion star* of W. Another star, $\mathcal{S}'$, is a companion star of W as well if $\mathcal{S} \subset \mathcal{S}'$. The problem that a companion star is not uniquely determined by W (and, consequently, by Q) will be investigated in the subsequent section.

We can conclude from $W = g(\tilde{W})$ that ∂W is covered by the members of $\{\boldsymbol{x} + \partial\tilde{W} : \boldsymbol{x} \in \mathcal{S}^\perp\}$. We consider exclusively the case where ∂W is *simply covered* in the sense that $A_{\boldsymbol{x}} := \partial W \cap (\boldsymbol{x} + \partial\tilde{W})$, ${}^\forall\boldsymbol{x} \in \mathcal{S}^\perp$, are connected. Then $A_{\boldsymbol{x}}, \boldsymbol{x} \in \mathcal{S}^\perp$, is an empty set, a point or a topological interval, which is called an *arc* because ∂W is a t-circle. A shell of $\mathcal{S}^\perp$ is called a *peripheral shell* if $A_{\boldsymbol{x}} \neq \emptyset$ with $\boldsymbol{x}$ being a member of the shell. In particular, it is called a *border shell* if $A_{\boldsymbol{x}}$ is an arc. The union, $\mathcal{P}$ (resp. $\mathcal{B}$), of all the peripheral (resp. border) shells of $\mathcal{S}^\perp$ is called the *peripheral* (resp. *border*) *subset* of $\mathcal{S}^\perp$; $\mathcal{B} \subset \mathcal{P}$. It follows that ∂W is covered by arcs in $\{A_{\boldsymbol{x}} : \boldsymbol{x} \in \mathcal{B}\}$, so that

$$\partial W \subset \Psi(\partial W) = \bigcup_{\boldsymbol{x}\in\mathcal{B}} (\boldsymbol{x} + \partial\tilde{W}), \tag{15}$$

where Ψ is the conjugate of an SSM specified by the pair $\{\mathcal{B}, \sigma\}$. This is the fundamental equation for ∂W. It will be used for a determination of ∂W. In this respect, a finite subset X of ∂W is called a *quasi-fixed set* of Ψ if $X \subset \Psi(X)$.

We confine our argument to the case where the set map, f, above is an SSM specified by the pair $\{\mathcal{S}, \sigma\}$. Then W is the fixed point of the SSSS $\Phi^\perp$: $W = \Phi^\perp(W) = \mathcal{S}^\perp + \tilde{W}$, so that (cf (3))

$$W = \bigcup_{\boldsymbol{x}\in\mathcal{S}^\perp} (\boldsymbol{x} + \tilde{W}), \qquad \tilde{W} = \bigcap_{\boldsymbol{x}\in\mathcal{S}^\perp} (-\boldsymbol{x} + W). \tag{16}$$

Then the fundamental equation for ∂W is rewritten as

$$\partial W \subset \Phi^\perp(\partial W) = \bigcup_{\boldsymbol{x}\in\mathcal{S}^\perp} (\boldsymbol{x} + \partial\tilde{W}), \tag{17}$$

which follows from (15) and $\mathcal{B} \subset \mathcal{S}^\perp$. The inflation rule satisfied by Q is $Q = \Phi(Q)$; Q is a fixed point of the map Φ and has $\mathcal{S}$ as its subset. The set map, Φ, represents the very PIR (point inflation rule). Other members of the LI-class to which Q belongs are not fixed points of Φ but they have, in the same LI-class, their predecessors and successors with respect to Φ.

Using expression (16) for $\tilde{W}$ together with (15) and the fact that W and $\tilde{W}$ are t-disks, we can conclude that $\mathcal{S}^\perp$ included in this expression can be replaced by an appropriate substar $\mathcal{T}^\perp$ of $\mathcal{S}^\perp$; it is sufficient (but not necessary) that $\mathcal{B} \subset \mathcal{T}^\perp$. Then the deflation rule, $Q \mapsto \hat{Q}$, is determined by $\mathcal{T}$ in place of $\mathcal{S}$. It is not rare that $\mathcal{T}^\perp$ is not only a special shell but also an outermost shell of $\mathcal{S}^\perp$. Then a site of Q belongs to $\hat{Q}$ iff it has $\mathcal{T}$ as its shell. If, moreover, $\mathcal{T}$ is the innermost shell of $\mathcal{S}$, $\hat{Q}$ is the set of all the n-pronged vertices of Q. This is the case for the decagonal QL in figure 2.

In the case of a reducible $\mathcal{S}$, all the arguments above remain correct even if the host module, $L^\parallel$, is replaced by $M^\parallel := \langle\mathcal{S}\rangle (\subset L^\parallel)$. The resulting QL is given by $Q \cap M^\parallel$. That is, to adopt $L^\parallel$ as the host module has no ground when $\mathcal{S}$ is reducible. A reducible star causes a few complications including this, and we will deal henceforth with the irreducible case.

We have considered the case where the inflation rule for a proper SSQL is represented as a PIR based on a star. Conversely, a PIR based on a star yields a proper SSQL as long as the fixed point of the conjugate set map is a t-disc in $E^\perp$.

6. Interplay between windows and the companion stars

We take a QL produced by a PIR. A problem is that the PIR is not uniquely determined by the QL because there exists an arbitrariness in the choice of the star $\mathcal{S}$. We will investigate this problem. Since $\mathcal{B} \subset \mathcal{P}$, we obtain the equation (cf (15)):

$$\partial W \subset \bigcup_{\boldsymbol{x}\in\mathcal{P}} (\boldsymbol{x} + \partial \tilde{W}). \tag{18}$$

Let $\mathcal{C}$ be the complement of $\mathcal{P}$ in $\mathcal{S}^{\perp}$. Then the former of (16) is rewritten as $W = W_{\mathcal{P}} \cup W_{\mathcal{C}}$ with

$$W_{\mathcal{P}} := \bigcup_{\boldsymbol{x}\in\mathcal{P}} (\boldsymbol{x} + \tilde{W}), \qquad W_{\mathcal{C}} := \bigcup_{\boldsymbol{x}\in\mathcal{C}} (\boldsymbol{x} + \tilde{W}). \tag{19}$$

The set $W_{\mathcal{P}}$ includes an open annulus whose outer boundary coincides ∂W (the set $W_{\mathcal{B}}$ defined in parallel to $W_{\mathcal{P}}$ can violate the 'annulus condition'), while $W_{\mathcal{C}} \subset {}^{\mathrm{i}}W$ because W is a t-disc and $\partial W \cap (\boldsymbol{x} + \partial \tilde{W}) = \emptyset, {}^{\forall}\boldsymbol{x} \in \mathcal{C}$. Then the set $W_{\mathcal{P}}$ is not a t-disc unless $\sigma < 2$ but becomes a t-disc owing to covering by $W_{\mathcal{C}}$, and $\mathcal{C}$ is called the *covering subset* of $\mathcal{S}^{\perp}$. The form, i.e., the boundary, of W is only determined by $\mathcal{P}$. Note that the module $\langle\mathcal{P}\rangle$ can be a true subset of $L^{\perp}$; $\mathcal{S}^{\perp}$ in this case is irreducible because of $\langle\mathcal{C}\rangle = L^{\perp}$.

The host module $L^{\parallel}$ and the scaling ratio σ are fixed for the present. Let W be a proper window and Ξ_W the set of all the companion stars of W. Then, for $\mathcal{S} \in \Xi_W$, we can define another star by $\mathcal{S}_1 := \mathcal{S} \cup \tau^q \mathcal{T}, q \in \boldsymbol{N}$, so that $\mathcal{S}_1 \in \Xi_W$, where $\mathcal{T} (\subset L^{\parallel})$ is a shell; this is realized if q is sufficiently large because $\tau^{-q}\mathcal{T}^{\perp}$, the conjugate of $\tau^q\mathcal{T}$, belongs then to the covering subset of $\mathcal{S}_1^{\perp}$. Note that the size of $\mathcal{S}_1$ can be arbitrarily large. The set Ξ_W is an infinite set. Since each member of Ξ_W is composed of a finite number of shells included in $Q = Q(W)$, Ξ_W includes a member whose radius is the smallest. Such a member is not necessarily unique but the set, Π_W, of all such members in Ξ_W is finite. Since Π_W is closed with respect to the union operation ($\cup$), Π_W has a maximal member. We call it a *canonical star*, which is uniquely determined by W and, consequently, by Q. We can adopt it as a representative member of Ξ_W. Similarly, we can define a *minimal star*, though such a star is not necessarily unique.

It is usually very difficult to determine the boundary ∂W of a fractal window W. It is possible, however, to determine easily a certain kind of points on ∂W. A shell $\mathcal{F}$ of $\mathcal{S}^{\perp}$ is called a *front shell* if $\rho\mathcal{F} \subset \partial W$ with $\rho := \sigma/(\sigma - 1) (\in \boldsymbol{Q}[\tau])$. Let $\boldsymbol{y} \in \mathcal{F}$ and $\boldsymbol{x} := \rho\boldsymbol{y} (\in \rho\mathcal{F})$. Then $\boldsymbol{x} \in \Phi^{\perp}(-\boldsymbol{x})$ because $\boldsymbol{y} + \sigma^{-1}\boldsymbol{x} = \boldsymbol{x}$, and

$$\rho\mathcal{F} \subset \Phi^{\perp}(\rho\mathcal{F}) \cap \partial W (\subset \partial W). \tag{20}$$

That is, $\rho\mathcal{F} (\subset \mathcal{R}^{\perp})$ is a quasi-fixed set of $\Phi^{\perp}$. It can be shown easily that the outermost shell of $\mathcal{S}^{\perp}$ is a front shell. The outermost Δ- (resp. Σ-) shell is a front shell as well if it is a border shell. A subset $\mathcal{F}$ of $\mathcal{S}^{\perp}$ is called a *front set* if it is composed of front shells. Then $\rho\mathcal{F}$ is a quasi-fixed set of $\Phi^{\perp}$ as well. The union of all the front shell(s) is a front set, which is a subset of $\mathcal{P}$. Note that a star yields a singular window if it has a front shell $\mathcal{F}$ such that $\rho\mathcal{F} \subset L^{\perp}$. For a front set, $\mathcal{F}$, a polygon, W_0, is defined by the condition $\rho\mathcal{F} = \mathcal{V}(W_0)$, where the symbol $\mathcal{V}(*)$ stands for the set of all the vertices of polygon *. We call W_0 the *induced polygon* (IP) by $\mathcal{F}$.

A crossing point of ∂W with a mirror axis of the type Δ is called a Δ-*point* of ∂W. A Σ-*point* is defined similarly. The regular *n*-gon whose vertices are the Δ-points (resp. the Σ-points) is denoted as P_Δ (resp. P_Σ), which is written as Δ_α (resp. Σ_β). A $2n$-gon whose vertices are composed of the Δ-points and the Σ-points is denoted as $P_{\Delta\Sigma}$, which may not be convex. The polygon, P_Δ (resp. P_Σ), is the IP of the outermost Δ- (resp. Σ-) shell provided that the shell is a front shell. On the other hand, $P_{\Delta\Sigma}$ is the IP of the union of the outermost Δ-shell and the outermost Σ-shell provided that they are front shells.

7. The von Koch island

In order to study the boundary ∂W of the window W, it is convenient to divide it into a finite number of arcs. We assume that the division keeps the point symmetry of W. A junction between two adjacent arcs in a division is called the *node* of the division. The division defines naturally two sets, namely, a G-symmetric point set $\Gamma(\subset \partial W)$ composed of all the nodes and the set $\mathcal{A}$ of all the arcs, and it is specified by the pair $\langle\langle\Gamma, \mathcal{A}\rangle\rangle$. In fact, it is only specified by Γ, which we call a *corral* of W. It is convenient to identify different members of $\mathcal{A}$ if they are equivalent with respect to $H^{\perp}(\simeq C_n)$. Then $r := \sharp(\mathcal{A})$ is called the *rank* of the division. It is sometimes convenient to identify $\mathcal{A}$ with an alphabet. Then ∂W is a 'loop word' on $\mathcal{A}$. The set $\mathcal{A}$ generates a free semi-group, and a word is an element of the semi-group.

Equation (15) shows that ∂W is covered by copies of $\partial\tilde{W}$. Different copies can cross. If a crossing point is located on ∂W, the same point on each of the two copies can be pulled back to a point on ∂W via a natural correspondence between ∂W and $\partial\tilde{W}$. Such a point is called a *primary node* of ∂W. We assume that Γ includes all the primary nodes by a reason to be presented shortly.

Let $\langle\langle\Gamma, \mathcal{A}\rangle\rangle$ and $\langle\langle\Gamma', \mathcal{A}'\rangle\rangle$ be two divisions of ∂W. Then the latter is called a *refinement* of the former if $\Gamma \subset \Gamma'$. The refinement relationship is represented as $\langle\langle\Gamma, \mathcal{A}\rangle\rangle \prec \langle\langle\Gamma', \mathcal{A}'\rangle\rangle$. Then all the arcs in $\mathcal{A}$ are words on $\mathcal{A}'$. The division $\langle\langle\Gamma, \mathcal{A}\rangle\rangle$ of ∂W is mapped to a division $\langle\langle\tilde{\Gamma}, \tilde{\mathcal{A}}\rangle\rangle$ of $\partial\tilde{W}$ with $\tilde{\Gamma} := \sigma^{-1}\Gamma$ and $\tilde{\mathcal{A}} := \sigma^{-1}\mathcal{A}$, which is a set of miniature arcs, $\tilde{A} := \{\sigma^{-1}A :^{\forall} A \in \mathcal{A}\}$. We include here (15) in the refinement procedure for $\langle\langle\Gamma, \mathcal{A}\rangle\rangle$; equation (15) shows that ∂W is covered by miniature arcs. A division is called a *complete division* (CD) and the relevant corral a *complete corral* (CC) if, in (15), ∂W is a loop word on $\tilde{\mathcal{A}}$ as well and every letter in the loop word ∂W on $\mathcal{A}$ is a word on $\tilde{\mathcal{A}}$. If ∂W has a CD, W is called *von Koch island* (vKI). The reason for the naming 'von Koch island' will become clear later on. Necessary conditions for a division of ∂W to be complete are (i) Γ is a quasi-fixed set of Ψ and (ii) Γ includes all the primary nodes of ∂W. A division is not a CD because the condition (i) is violated. If this is the case, a miniature arc is divided by a point of Γ in the covering (15). The point can be pulled back to a dividing point on the relevant arc of ∂W. Such a point is called a *secondary node* of ∂W. All the secondary nodes must be included in Γ to obtain a CD. The resulting division may not be a CD by the same reason. Then this *pull back procedure* must be repeated until we arrive at a CD though it may not stop in a finite number of steps.

Let W be a vKI and $\langle\langle\Gamma, \mathcal{A}\rangle\rangle$ its CD. Then $A \in \mathcal{A}$ has its own word, X_A, on $\tilde{\mathcal{A}}$. The set of words, $\{X_A :^{\forall} A \in \mathcal{A}\}$, defines the relationship between letters in the two generations of alphabets; this relationship is called the *substitution rule* (SR). A letter in $\mathcal{A}$ and its counterpart in $\tilde{\mathcal{A}}$ are not distinguished when the SR is regarded as a symbol dynamics. The map, ϕ, associated with the symbol dynamics is defined by $X_A = \phi(A),^{\forall} A \in \mathcal{A}$. The SR is represented geometrically as

$$A = \bigcup_{B\in X_A} F_{AB}(B), \qquad {}^{\forall}A \in \mathcal{A} \tag{21}$$

with $F_{AB} \in \mathcal{G}_{\sigma}^{\perp}$. The homogeneous part of the affine transformation, F_{AB}, takes the form $\sigma^{-1}R^i$, which is specified by the angle number, i. If a single letter appears multiply in the right-hand side of (21), the form of the affine transformation assigned to the letter depends on the order of the letter in the word, X_A. Equation (21) shows that $\mathcal{A}$, the set of arcs, is a fixed point of an SSSS [21], and each letter in $\mathcal{A}$ is a von Koch curve; ∂W is a cyclic succession of such curves. We may say that the present QL is of the *von-Koch type*. Hereafter we consider exclusively QLs of this type. It is very likely that a QL is of this type if it is MLD with another QL of this type. Then an MLD class is characterized by an alphabet $\mathcal{A}$ and the relevant SR.

The rank r is one of the characteristics of the SR. If $r = 1$, we may say the vKI to be uniform because all the arcs of ∂W are of the same kind. In the case of a uniform vKI, a single letter, A, is concerned, and the word substituting A takes the form, $A^m, {}^\exists m \in \boldsymbol{N}$.

Each arc of ∂W is not self-similar if the vKI is not uniform but we may say that the arcs are 'self-similar' as a set. The fractal dimension (Hausdorf dimension) of ∂W is given by $D = \ln\lambda/\ln\sigma$, with $\lambda(>1)$ being the Perron–Frobenius eigenvalue of the incidence matrix M associated with the SR (i.e., the map, ϕ). The Perron–Frobenius eigenvalue of a uniform is m.

Symmetry consideration for letters and words is important because QLs under consideration have high point symmetries. The mirror image of a letter (or word) X is denoted as $\bar{X}$; $X = \bar{X}$ if it is achiral (mirror symmetric). If $X = ABC$, for example, we obtain $\bar{X} = \bar{C}\bar{B}\bar{A}$. The word associated with an achiral letter is palindrome-like. Let c (resp. a) be the number of chiral (resp. achiral) pairs of letters in $\mathcal{A}$. Then $r = 2c + a$. It is convenient to order the r letters in $\mathcal{A}$ so that the kth letter and the $(c + a + k)$th letter form a chiral pair for $1 \leqslant k \leqslant c$. Then the last c are the mirror images of the first c and the middle a are achiral. These letters are represented in this order as $A, B, C, \ldots$. Note, however, that the symbol A is often used as a symbol for a generic letter.

The incidence matrix M of an SR has a certain symmetry with respect to the chirality. Then a reduced incidence matrix M' is naturally defined such that its dimension is $r' := c+a$; M' is obtained if every chiral letter and its mirror image are not distinguished when counting letters in a word. We call r' the reduced rank of the SR. Note that $c = r - r'$ and $a = 2r' - r$. The Perron–Frobenius eigenvalue is shown to be common between M and M'. If an SR with rank 2 has two letters forming a chiral pair, the reduced rank is 1. Then we may say that the relevant vKI is *quasi-uniform*. A sufficient condition for a vKI to be uniform (resp. quasi-uniform) is that the corral of the relevant CD satisfies $\Gamma = P_\Delta$ or $\Gamma = P_\Sigma$ (resp. $\Gamma = P_{\Delta\Sigma}$).

The word associated with a letter in an SR specifies only the order of the letters, and we must append it a list of the angle numbers in order to geometrically specify the SR. Such a list for a seven-letter word takes, for example, the form, $\{0120\bar{2}\bar{1}0\}$ with $\bar{i} := -i$. This example is an anti-palindrome in the sense that the signs are reversed when it is read reversely. A list of the angle numbers has this property if the word is associated with an achiral letter.

Let $\langle\langle\Gamma, \mathcal{A}\rangle\rangle$ be a CD of ∂W and assume that another CD, $\langle\langle\Gamma', \mathcal{A}'\rangle\rangle$, is a refinement of the former. Then we may say that the two SRs are equivalent if the two sets $\mathcal{A}$ are $\mathcal{A}'$ are different generator sets of a single semi-group. If this is the case, every member of $\mathcal{A}$ is a word on $\mathcal{A}'$ and, moreover, the two SRs are translated to each other. Then the Perron–Frobenius eigenvalues associated with the two SRs are the same. If $\mathcal{A}$ of a CD includes one or several achiral letters (arcs), we can obtain its refinement CD by halving all the achiral letters (arcs). In some case, halving procedure can be restricted to achiral letters selected appropriately. If P_Δ (resp. P_Σ) is not a subset of Γ with $\langle\langle\Gamma, \mathcal{A}\rangle\rangle$ being a CD, $\mathcal{A}$ includes an achiral letter (arc) bisected by a Δ-point (resp. a Σ-point). This bisection yields a second SR which is equivalent to the original. Therefore, every vKI has a CD such that $P_\Delta \subset \Gamma$ or $P_\Sigma \subset \Gamma$. Moreover, every vKI has a CD such that $P_{\Delta\Sigma} \subset \Gamma$. We can restrict our argument to special CDs satisfying one of these three conditions.

The window W of a QL is not reconstructed only by information on the SR for $\mathcal{A}$; we need information on the loop word representing ∂W. Since ∂W has the n-fold point symmetry, the loop word can be divided into n identical words as F^n with F being a $2\pi/n$-sector of ∂W; the nodes of the division form P_Δ or P_Σ. The word F, which is an achiral arc of ∂W, is called a *fundamental arc* (FA) of ∂W. If $P_{\Delta\Sigma} \subset \Gamma$, we may write $F = H\bar{H}$, where H is usually a chiral word, which we call a *half FA*.

In order to determine W, we need to obtain a CD of ∂W and, consequently, to know the primary nodes and the secondary ones of ∂W. However, they depend on the structural details of W. It is desirable that the CD and the relevant SR are obtained in some way prior to the determination of W. A partial but practical answer to this question is presented in the subsequent section.

The theory developed up to present in this section is not restricted to QLs produced by the PIR. We assume it hereafter. Let $\langle\langle\Gamma_0, \mathcal{A}_0\rangle\rangle$ be a CD and $\langle\langle\Gamma_1, \mathcal{A}_1\rangle\rangle$ its refinement, where $\mathcal{A}_1 = \sigma^{-1}\mathcal{A}_0$. Then we obtain $\Gamma_1 = \Phi(\Gamma_0) \cap \partial W$ because of (17). More generally, an increasing series of CCs, $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$, is defined by

$$\Gamma_{j+1} = \Phi^{\perp}(\Gamma_j) \cap \partial W \subset \Phi^{\perp}(\Gamma_j). \tag{22}$$

It yields a series of CDs of ∂W, $\{\langle\langle\Gamma_j, \mathcal{A}_j\rangle\rangle : j \geqslant 0\}$, such that $\langle\langle\Gamma_j, \mathcal{A}_j\rangle\rangle \prec \langle\langle\Gamma_{j+1}, \mathcal{A}_{j+1}\rangle\rangle$. Remember that Γ_j is cyclically ordered because $\Gamma_j \subset \partial W$. The limiting set, Γ_∞, is dense on ∂W because $\Gamma_j \subset \Gamma_\infty \subset \partial W$ and the sizes of miniature arcs become infinitesimals as j goes to infinity. The descending series of alphabets, $\{\mathcal{A}_j : j \geqslant 0\}$, can be naturally extended backward ($j < 0$). If ∂W is exactly divided into letters in a past generation, we can redefine them as the zeroth generation letters. In this case, some of the letters may not be used as the building blocks of ∂W. Even if this is the case, all the letters are necessary to make the SR closed. It is in a sense more economical to use letters in an earlier generation as the building blocks.

8. Proto-von-Koch-island

Since Γ_j is cyclically ordered, a zigzag loop is obtained by drawing a bond between every successive pair of its members. It defines a G-symmetric polygon, W_j, which we call a *proto-von-Koch-island* (PvKI) of the jth generation (see figure 3). It follows that $\mathcal{V}(W_j) \subset \Gamma_j$.[2] If $\mathcal{V}(W_j) \neq \Gamma_j$, Γ_j includes a point which is not located on any vertex of W_j but on an edge of W_j. We call this point a *virtual vertex* of W_j, and Γ_j a *corral* of W_j; $\Gamma_j \subset \partial W_j$. A PvKI is regarded to be an approximant to the vKI. The series of PvKIs, $W_0, W_1, W_2, \ldots$, converges on the window, W.

The SSSS defined by (21) allows us an iterative solution for $\mathcal{A}$. It converges on $\mathcal{A}$, the attractor, if the initial set $\mathcal{A}_{\mathrm{initial}}$ includes a non-empty member [21]. To each arc, $A \in \mathcal{A}$, we can assign the chord, i.e. a shortcut combining both the ends, $\underline{A}$. If the resulting set $\mathcal{Z}$ of chords, i.e., intervals, is used as $\mathcal{A}_{\mathrm{initial}}$ for the iterative solution, we obtain a set $\mathcal{Z}^{(j)}$ of open zigzag curves after jth iteration; $\underline{A}^{(j)} \in \mathcal{Z}^{(j)}$ is nothing but an arc of the approximant vKI, W_j, and its chord is common to that of A. In this respect, we must not confuse the zigzag curve, $\underline{A}^{(j)}$, with the miniature chord, $\sigma^{-j}\underline{A}$; $\underline{A}^{(j)}$ is a succession of such miniature chords, and converges on the arc A (see figure 3).

A PvKI as well as a CD of ∂W is not uniquely determined by a vKI. We can exploit it to obtain a simpler PvKI. For this purpose, directions of edges of a PvKI are classified into equivalence classes with respect to the point group C_n. The *direction number d* defined as the number of the equivalence classes is one of the characteristics of the PvKI. A PvKI with a smallest d is in a sense simplest, though it may not necessarily have a priority among PvKIs of a vKI. *Uni-directional PvKI*s, for which $d = 1$, are divided into the type Δ group and the

[2] The symbol $\mathcal{V}(*)$ defined in the paragraph including (20) will be used frequently in this section.

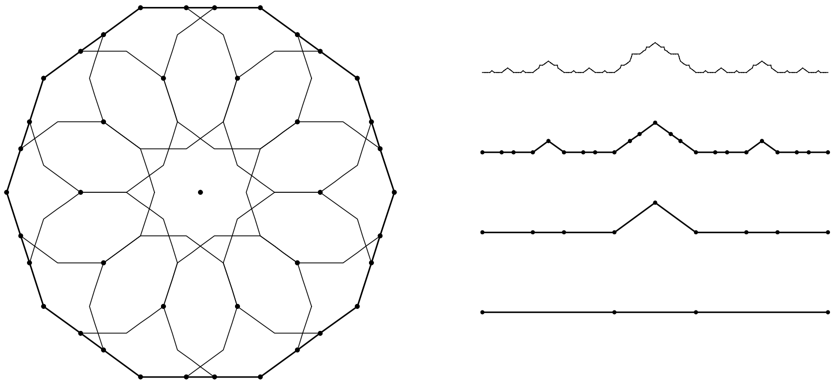

Figure 3. Left: the regular decagon drawn in thick lines is a PvKI, W_0. Smaller ten decagons which overlap each other are copies of the miniature PvKI, whose scale is $\sigma^{-1}(=\tau^{-2})$-times the original. The corral of the PvKI is formed of the dots on the boundary. Right: the line at the bottom shows an edge of the PvKI on the left. The SR for the intervals of the PvKI is $A \to ABA$ and $B \to AA$ with A being the longer interval. The next two zigzag lines are the first two results of successive applications of the SR to the bottom line. The resulting von Koch curve is shown on the top.

type Σ group by the directions of their edges. Note that a uniform PvKW is not necessarily uni-directional.

If the CD associated with a PvKI is a refinement of that with another PvKI, we may say that the former PvKI is a *refinement* of the latter. A PvKI is minimal if it is not a refinement of any other PvKI. The PvKI in figure 3, for example, is not minimal. It has 20 virtual vertices in addition to ten real vertices. They are the primary nodes, and the dodecagon defined by these nodes is a minimal PvKI, which has the original PvKI as its refinement. However, this PvKI is not necessarily simpler than the original because it is bi-directional.

A PvKI is defined on the basis of a CD of ∂W. It is desirable that it is obtained prior to the determination of W. In this respect, an IP, W_0, by a front set of $\mathcal{S}^\perp$ is a promising candidate for a PvKI of W. A CD of ∂W is obtained if a CC Γ_0 is appended to W_0, where $\Gamma_0 \subset \partial W_0$. To obtain the CC, we assume that a series of windows, $\{W_j : j \geqslant 0\}$, generated by the set map (4) are G-symmetric polygons. Since we have $W_j = (\Phi^\perp)^j(W_0)$, a vertex of W_j is a vertex of a copy of the miniature polygon, $\sigma^{-j}W_0$, or a crossing point between edges of two copies. If the latter contribution is absent for all j, $\Gamma_0 := \mathcal{V}(W_0)$ is a CC. If the latter contribution is present for $j = 1$, two edges belonging to different miniature polygons divide the relevant two edges (see figure 3). Each of the two dividing points can be pulled back to a point of the relevant edge of W_0. In a favorable case, this point is a primary node of ∂W. A secondary node is obtained, if necessary, by a similar way. Then W_0 can be a PvKI of W if the nodes of these types are appended to the corral Γ_0 of W_0. Since W_0 is an IP, we obtain $\mathcal{V}(W_0) \subset \mathcal{R}^\perp$, and, consequently, $\Gamma_0 \subset \mathcal{R}^\perp$. Then $K^\perp := \langle\Gamma_0\rangle$ is a fractional ideal of $L^\perp$, and Γ_j is a finite G-subset of $K^\perp$, which we may call the *support module* of Γ_j s. The present heuristic method of obtaining a PvKI is found to be often successful.

The case where P_Δ, P_Σ or $P_{\Delta\Sigma}$ is a PvKI is of a particular importance. The PvKI of the type P_Δ or P_Σ is uni-directional but bi-directional for a generic PvKI of the type $P_{\Delta\Sigma}$; the PvKI of the type $P_{\Delta\Sigma}$ is uni-directional iff it is a stellated n-gon (a non-convex $2n$-gon). If a vKI has a PvKI, W_0, of one of these three types, its FA (or half FA) is naturally defined by taking $\mathcal{V}(W_0)$ as the set of the nodes of the division, and an edge of W_0 is the chord of the

FA. If the PvKI is of the non-uniform type, the edge is divided by virtual vertices into a finite number of intervals (see figure 3), which are associated with the basic arcs forming the FA. Then the relative lengths of the intervals are necessary to specify the PvKI. Their list is an important characteristic of the PvKI.

The area $|W|$ of W determines the site density of $Q(= Q(W))$. The difference $|W|-|W_0|$ is a sum of the increments due to replacement of the intervals forming ∂W_0 by the relevant arcs. Then a row vector $\boldsymbol{D}$ formed of the increments is naturally defined. A similar row vector, $\boldsymbol{D}_0$, to $\boldsymbol{D}$ is defined with respect to the relationship between W_0 and W_1. Using the recursive nature of the structure of $\mathcal{A}$, we can show that[3]

$$\boldsymbol{D} = \boldsymbol{D}_0(I - \sigma^{-2}M)^{-1}. \tag{23}$$

Since $|W_0|$ and $\boldsymbol{D}_0$ are calculated exactly by elementary geometry, so is $|W|$. If W_0 is a uniform PvKI, we obtain $|W| = |W_0| + \sigma^2(\sigma^2 - m)^{-1}(|W_1| - |W_0|)$.

Equation (22) together with the relationship $\mathcal{V}(W_j) \subset \Gamma_j$ indicates that $W_{j+1} = \Phi^{\perp}(W_j)$, $j \geqslant 0$. We call W_0 a *strong* PvKI if this is the case. Then the series $\{W_j : j \geqslant 0\}$ is increasing (resp. decreasing) according as $W_0 \subset W_1$ (resp. $W_0 \supset W_1$), which follows from the monotonicity of the map $\Phi^{\perp}$. Then W_0 inscribes (resp. circumscribes) W. If ∂W_j is compared with ∂W_0, a zigzag line replacing an interval of ∂W_0 detours outside (resp. inside) of W_0. It crosses the interval in a generic case. All the components of $\boldsymbol{D}$ are positive (resp. negative) for the increasing (resp. decreasing) case but both the signs appear in a generic case.

9. Isomerism

The point group of QLs (SSQLs) is fixed in this section to one of the three. We define two QLs to be *isomeric* if the window of one of them has a geometrically similar arc to that of the other or, equivalently, their windows are characterized by a common SR. We can classify QLs with respect to the isomerism. Two QLs are isomeric if they are MLD, and an isomeric class is divided into a number of MLD classes. Two QLs belonging to an isomeric class can have windows which are geometrically similar to each other. Then we may say that they are *strongly isomeric* (s-isomeric). Two QLs are s-isomeric iff the peripheral subsets (PSs) of the relevant two stars are similar. We can classify QLs with respect to the s-isomerism; an isomeric class is divided into *s-isomeric classes*. Note, however, that s-isomerism is not compatible with the MLD.

We fix σ in the rest of this section. Two QLs belonging to a common *s-isomeric class* are geometrically similar to each other iff the relevant two windows satisfy $W = \sigma^q W', {}^{\exists} q \in \boldsymbol{Z}$, or, equivalently, the relevant two PSs satisfy $\langle\mathcal{P}\rangle = \langle\mathcal{P}'\rangle$. Then the members of an s-isomeric class, $\mathcal{I}$, can be classified with respect to the similarity relationship, $\sim$; the equivalence class to which a given QL belongs is determined by the module $\langle\mathcal{P}\rangle$ with $\mathcal{P}$ being the relevant PS. We use hereafter the symbol $\mathcal{I}$ for $\mathcal{I}/\sim$. Then there exists a bijection between $\mathcal{I}$ and the set of all the G-submodules of $L^{\parallel}$, and $\mathcal{I}$ is countably infinite. Since different members of $\mathcal{I}$ are not MLD with each other, an isomeric class includes an infinite (but countable) number of MLD classes.

Let $M^{\parallel}$ be the submodule specifying a member of $\mathcal{I}$. Then the number $(L^{\parallel} : M^{\parallel})$ gives a measure of complexity of the member. The simplest member is that for which $M^{\parallel} = L^{\parallel}$ and is called the *principal QL*. The next simplest member is that for which $M^{\parallel} = \langle\Sigma\rangle$ (unless $n = 12$) and is called the *quasi-principal QL*. The window W' of the quasi-principal QL is related to the window W of the principal QL by $W' = \omega W$, which differs in size and in orientation

[3] If $r' < r$, we obtain a simpler expression, $\boldsymbol{D}' = \boldsymbol{D}_0'(I - \sigma^{-2}M')^{-1}$, where $\boldsymbol{D}'$ and $\boldsymbol{D}_0'$ are r'-dimensional row vectors obtained by a truncation from $\boldsymbol{D}$ and $\boldsymbol{D}_0$, respectively.

from W. Correspondingly, the relevant PSs are related by $\mathcal{P}' = \omega\mathcal{P}$. The window of a generic member of $\mathcal{I}$ is written as νW or $\nu W'$ with $^\exists\nu \in \boldsymbol{Z}^\times[\tau]$ and $|\nu\bar{\nu}| > 1$, and the relevant PS is written as $\nu\mathcal{P}$ or $\nu\mathcal{P}'$. Let $\mathcal{S}^\perp$ be the star specifying the principal QL. Then $\omega\mathcal{S}^\perp$, $\nu\mathcal{S}^\perp$ and $\nu(\mathcal{S}')^\perp$ are reducible, and cannot be the companion stars of the relevant windows. Each of these three stars can be changed to an irreducible star if a shell of the form $\tau^{-q}\Delta$ with q being a sufficiently large integer is appended to $\omega\mathcal{C}$, $\nu\mathcal{C}$ or $\nu\mathcal{C}'$ to obtain the covering part paired with the relevant PS.

10. Computer-aided drawing of QLs

Let Φ be the set map of a PIR. Then the QL determined by the PIR is given by $Q = Q(W)$ with W being the fixed point of $\Phi^\perp$. The expression for $Q(W)$ in the projection method is given by (6). A programming for testing the condition $\boldsymbol{x}^\perp \in W$ imposed on $\boldsymbol{x}^\parallel \in Q(W)$ is easy for the case of a polygonal window but is almost impossible for the case of a fractal window in which we are interested. Therefore, we must approach to $Q(W)$ via a different route from the projection method.

If W is a vKI, there exists a series of PvKIs, $\{W_j : j \geqslant 0\}$, converging on W. The series yields a series of QLs, $\{Q_j : j \geqslant 0\}$ with $Q_j = Q(W_j)$, converging on $Q(W)$. The window W_j is regarded to be the jth *polygonal approximant* to W, and Q_j to be the jth *approximant* to Q. The boundary of W_j becomes increasingly complicated with j, and so does the programming for Q_j. We consider first the case of a strong PvKI, where $W_{j+1} = \Phi^\perp(W_j)$, and $Q_{j+1} = \Phi(Q_j)$. A patch of the zeroth approximant $Q_0 = Q(W_0)$ is calculated easily with a computer because W_0 is a simple polygon. Since the equation $Q_{j+1} = \Phi(Q_j)$ is algorithmic, patches of subsequent approximants are recursively obtained with the aid of a computer. If the size of a central patch of Q is specified, the patch is obtained after a finite number of iterations. This provides us with a way for computer-aided drawing of Q. This way, in fact, remains correct even if W_0 is not a strong PvKI as long as the QL is produced by a PIR; the PIR with the starting QL, $Q(W_0)$, yields another series $\{Q'_j : j \geqslant 0\}$ converging on $Q(W)$. The series $\{W'_j : j \geqslant 0\}$ of the relevant windows is generated by $W'_{j+1} = \Phi^\perp(W'_j)$, $j \geqslant 0$, but W'_j s are not necessarily polygons (t-discs).

The starting set in the above-mentioned iterative procedure of obtaining an QL is an infinite set. The same QL or another QL belonging to the relevant LI class is generated from a finite set chosen appropriately. If this is the case, we call the set a *germ* of the LI class. The global point symmetry of the QL is the same as that of the germ. If $\{0\}$, the set composed of the origin alone, is a germ of an LI class, we obtain

$$Q(W) = \mathcal{S}_\infty. \tag{24}$$

This does not proves, however, that the star in the jth generation, $\mathcal{S}_j$, is a patch of Q; some sites of Q may be absent in a peripheral region of $\mathcal{S}_j$ because they come on the inflation from other sites than those included in $\mathcal{S}_{j-1}$. A necessary but not sufficient condition for the set $\{0\}$ to be a germ of an LI class is that W, the window centered on the origin, is regular. This subject will be discussed in specific examples in the subsequent section.

11. Examples

We deal with canonical stars except the case stated otherwise. Nine QLs produced by PIRs are selected to exemplify various situations to appear in an application of the PIR. Characteristics of these QLs are summarized in table 1. A QL in table 1 will be referred henceforth to by its number in the table. The stars of the three of the nine QLs are singly shelled. The vKI,

Table 1. This table lists characteristics of nine QLs produced by PIRs. It is divided into two parts, the lower of which is continued to the upper. The windows of the QLs are vKIs, which are of the strong type except the case of no. 7. The top of each column shows the attribute of the column. The attributes shown by symbols are n: the point symmetry, p: the power p in $\sigma = \tau^p$, $\mathcal{S}$: the star, $r(r')$: the rank (the reduced rank) of the PvKI, λ: the Perron–Frobenius eigenvalue and D: the fractal dimension. The word ABC in the second column is a half FA. The symbol $\mathcal{S}_{\rm sub}$ used in the fifth column and eighth stands for the star $\{1, \sqrt{2}, \tau; \tau^{-1}, 1, \sqrt{2}\}$ while $\mathcal{S}^{(5)}$ and $\mathcal{S}^{(6)}$ for the stars nos. 5 and 6, respectively. The PvKW of the fifth column is one of the three induced polygons, P_Δ, P_Σ and $P_{\Delta\Sigma}$. The length list in the eighth column is only presented for the case of a non-uniform PvKI. The nomenclature for the letters in the seventh column follows the convention in section 7. The words in the seventh column are listed in the order, X_A, X_B and X_C. The word X_B of no. 2 stands for $\bar{B}\bar{A}ABC\bar{C}\bar{B}\bar{A}A$ but that of no. 6 for $B\bar{A}ABB\bar{A}AB$.

	n	p	$\mathcal{S}$	PvKI	(r')	FA	Length list
1	10	2	$\{1; \emptyset\}$	Δ	2	ABA	$\{\tau, 1, \tau\}$
2	10	3	$\{1, \tau; 1\}$	$\Delta\Sigma$	6 (3)	ABC	$\{1, 2\tau, 1\}$
3	10	3	$\{\tau^{-1}, 1, \tau; 1\}$	Δ	1	A	
4	8	1	$\{1; \emptyset\}$	Δ	3 (2)	$AB\bar{A}$	$\{1, \sqrt{2}, 1\}$
5	8	2	$\mathcal{S}_{\rm sub} \cup \{1 \vee 1\}$	Δ	1	A	
6	8	2	$\mathcal{S}^{(5)} \cup \{1 \vee \tau^{-1}\}$	Σ	3 (2)	$AB\bar{A}$	$\{1, \sqrt{2}, 1\}$
7	8	2	$\mathcal{S}^{(6)} \cup \Delta_{1/\tau} \cup \Delta_2$	Δ	1	A	
8	8	2	$\tau\mathcal{S}_{\rm sub} \cup \{\tau \vee 1, 1 \vee \tau\}$	Σ	1	A	
9	12	1	$\{1; \emptyset\}$	Δ	1	A	

Word	Angle number list	λ	D
ABA, AA	$\{000\}, \{1\bar{1}\}$	$1 + \sqrt{3}$	1.044
AB, X_B, BC	$\{00\}, \{00\bar{1}\bar{1}\bar{1}1110\}, \{00\}$	$2 + \sqrt{7}$	1.064
A^6	$\{012\bar{2}\bar{1}0\}$	6	1.241
AB, $\bar{A}A\bar{A}A$	$\{00\}, \{01\bar{1}0\}$	$\frac{1}{2}(1 + \sqrt{17})$	1.067
A^9	$\{012\bar{1}01\bar{2}\bar{1}0\}$	9	1.246
$AB\bar{A}A\bar{A}A$, X_B	$\{0001\bar{1}0\}, \{0011\bar{1}\bar{1}00\}$	$\frac{1}{2}(9 + \sqrt{17})$	1.067
A^9	$\{0\bar{1}\bar{2}\bar{1}01210\}$	9	1.246
A^7	$\{01\bar{1}01\bar{1}0\}$	7	1.104
A^4	$\{01\bar{1}0\}$	4	1.053

the conjugate star, $\mathcal{S}^\perp$, and the miniature window are shown in figure 4 for each of the three QLs. Similar figures for the remaining six are shown in figure 5. The (reduced) incidence matrices for the SRs are not listed in the table because they are easily obtained from the SRs. The decagonal QLs and the octagonal ones are discussed separately in the subsections below.

Every QL in table 1 is the principal QL in the relevant weak isomeric class, so that windows of an infinite number of QLs forming the isomeric class have been determined.

11.1. Decagonal QLs

A simplest case for $n = 10$ is $\sigma = \tau$ and $F_{\mathcal{S}} = \{1; \emptyset\}$. This is not listed in table 1. The star is singly shelled, and the relevant shell is Δ. The IP of the conjugate shell, $\Delta^\perp$, is P_Δ. It is the fixed point of the relevant set map; this is due to the inequality $\sigma < 2$. The relevant QL is the one associated with the (pentagonal) Penrose tiling which includes tiles of a regular pentagon, a pentagonal star, a boat and a thin rhombus. The present window is singular, and the Penrose tiling has no local centers with the ten-fold rotational symmetry.

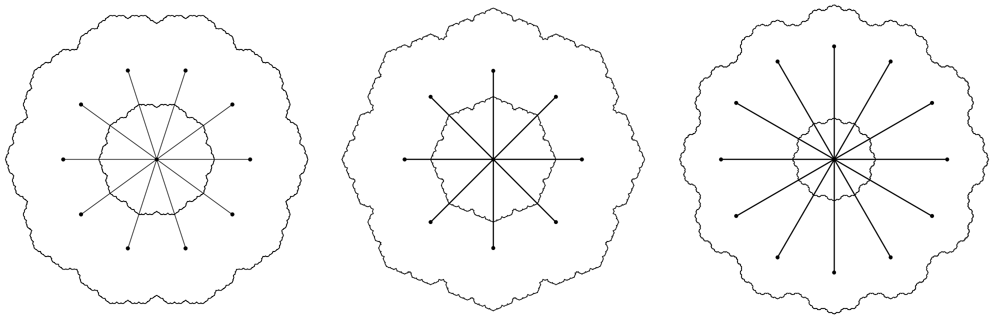

Figure 4. The vKI, the conjugate star, $\mathcal{S}^{\perp}$, and the miniature window are superposed for each of the following three QLs: no. 1(left), no. 4 (middle) and no. 9 (right). These stars are singly shelled.

A next simplest case is no. 1. The star is common to the example above but $\sigma = \tau^2$. The regular decagon in figure 3(left) is P_Δ, the IP of the shell. It is a PvKI, which is τ^{-1}-times the window of the Penrose tiling above; the QL with P_Δ as its window is a scaled version of the tiling by the ratio, τ. The vKI shown in figure 4(left) is circumscribed by the PvKI, and is a moth-eaten version of the PvKI. The fractal dimension, 1.044, of ∂W as listed in table 1 shows that ∂W is a mild fractal, which we can observe in figure 4(left). Since $\rho(= \tau)$ is integral, W is singular as well. If a pentagon cluster is used as a germ for the PIR, we obtain a QL as shown in figure 6(left). Since the vKI is barely smaller than the window of the Penrose tiling, the present QL is derived from the QL associated with the Penrose tiling by deleting a part of the sites, the ratio of which is estimated to be 4.4%. A quasiperiodic tiling associated with the present QL includes tiles of a hexagon in addition to basic tiles of the Penrose tiling. Every star in the Penrose tiling is surrounded by five pentagons and a part of stars are transformed in the present tiling to boats. More precisely, a pair of a star and a pentagon is transformed to a pair of a boat and a hexagonal tile by deleting one of the five concave vertices of the star. However, there is no local rule determining such a pair of a star and a pentagon because the two QLs are not MLD. The tiling has an SR, in which the decoration of every tile preserves the point symmetry of the tile. The SR of the present tiling is simpler in this respect than that of the Penrose tiling. We call the present tiling the *para-Penrose-tiling*; the original Penrose tiling is regarded to be the zeroth polygonal approximant to it.

The reason why a fractal window being close to a polygonal window appears for no. 1 is ascribed to the inequality $2 < \sigma < 3$ for the present $\sigma(= \tau^2)$, which can be understood by figure 3(left). By this reason, a star whose outermost shell is a special shell is likely to yield a QL which belongs to the common isomeric class to that of no. 1 if it is paired with this σ. Therefore, every decagonal QL with τ^2-scaling has its 'para-' version if its window is a regular decagon. There exist another three QLs whose windows are regular decagons [10], and their para-versions are isomeric to the para-Penrose-tiling.

We turn our attention to the decagonal QL no. 2. The star of the PIR is obtained from the leftmost star of figure 1 by deleting the innermost shell, and is self-conjugate. A central patch of the QL is shown in figure 6(right). The present PvKI, $P_{\Delta\Sigma}$, is the largest stellated decagon inscribed in W. In our table, it is the only example such that its corral includes the secondary nodes. A decagonal tiling is associated with the present decagonal QL. Its tiles include a decagonal tile and a tile of a hexagon in addition to the four basic tiles of the Penrose tiling; the decagonal tile includes an isolated site on its center. This is understood because the smallest regular decagon circumscribing W is the window of the Penrose tiling.

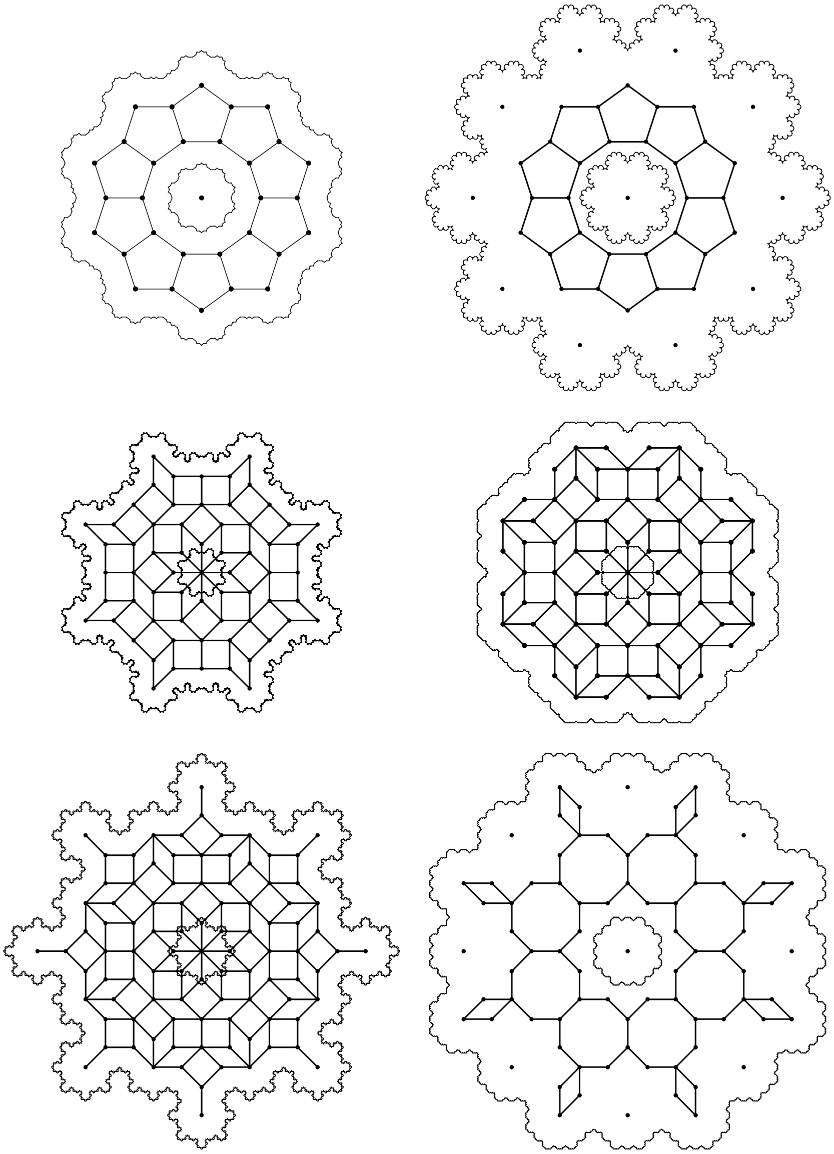

Figure 5. A similar figure to figure 4 but for nos. 2 (top left), 3 (top right), 5 (middle left), 6 (middle right), 7 (bottom left) and 8 (bottom right).

Decagonal QL no. 3 is exhibited already in figure 2. The relevant vKI is shown in figure 5(top right). The second PvKI, $P_{\Delta\Sigma}$, is uniform as well. The relevant SR is specified by the set of the angle numbers, $\{0\bar{2}1\bar{1}20\}$. The boundary of the present vKI is considerably wild in agreement with $D \approx 1.241$.

11.2. Octagonal QLs

A simplest PIR among those for octagonal QLs is that for the QL no. 4. The vKI shown in figure 4(middle) is not much different from the PvKI, P_{Δ}, which circumscribes it. Since

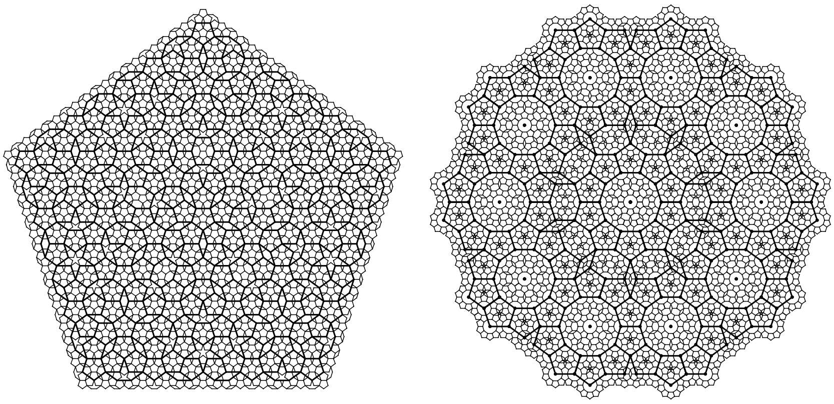

Figure 6. Left: a pentagonal patch of the QL no. 2 (thick lines) and its inflation (thin lines). The central pentagon is the germ of this patch. Right: a similar figure to the left but for the QL no. 3. The thick lines and thin lines show, respectively, the second generation motif and third one.

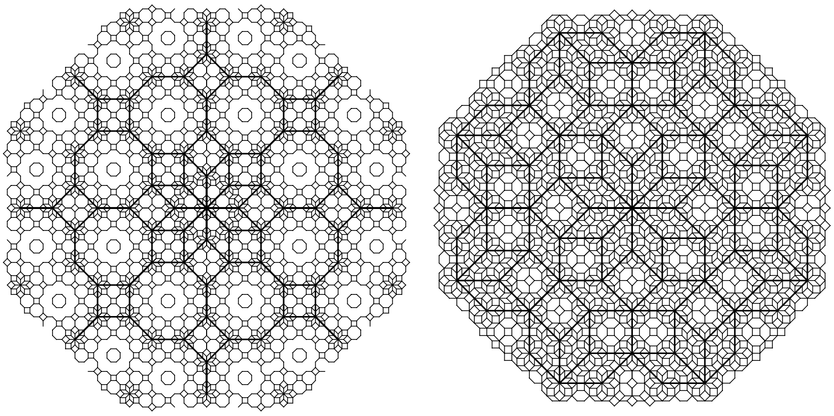

Figure 7. Octagonal patches of the QLs no. 7 (left) and no. 8 (right).

$\sigma(=\tau)$ satisfies the inequality $2 < \sigma < 3$, the isomeric class including the present QL includes a lot of QLs which have their counterparts in a group of octagonal QLs whose windows are regular octagons. For example, the star of the quasi-principal QL in this isomeric class is specified by $F_{\mathcal{S}} = \{1, \tau^{-1}\}$. The relevant window is regular but $\mathcal{S}_\infty$ does not agree with $Q(W)$; Δ is the first shell of $Q(W)$ but it is not included in $\mathcal{S}_\infty$.

More interesting octagonal QLs are nos. 7 and 8. A patch of the QL no. 7 is shown in figure 7(left). The window has two PvKIs, P_Δ and $P_{\Delta\Sigma}$. The second PvKI, $P_{\Delta\Sigma}$, is quasi-uniform. The star associated with the QL no. 8 is the third one in figure 1. It is not canonical. A patch of the QL is presented in figure 7(right). The QL includes tiles of a hexagon in addition to a square tile and a rhombic tile which are basic tiles of the Ammann–Beenker tiling. The tiling has an SR, in which the decoration of every tile preserves the point symmetry of the tile and the central part is occupied by a miniature of the original tile (the decoration rule for the hexagon tile is not shown). The relevant window, W, has P_Σ as its PvKI, which is the same regular octagon as the window of the Ammann–Beenker tiling. The window, W, shown in figure 5(bottom right) is barely smaller than P_Σ, which circumscribes W. The present QL is derived from the QL associated with the Ammann–Beenker tiling by deleting a part of the sites to yield hexagonal tiles; the ratio of the deleted sites is estimated to be 6.1%. We call the present tiling the para-Ammann–Beenker-tiling ; the original Ammann–Beenker tiling is regarded to be the zeroth polygonal approximant to it. Note that no seven-pronged vertex is included in the former tiling in contrast to the latter. It should be mentioned that the second QL discussed in the beginning paragraph deserves to be called a para-Ammann–Beenker-tiling as well because its window is another moth-eaten version of the window of the Ammann–Beenker tiling.

We cannot present the details of the two QLs, nos. 5 and 6, because of the limitation of the space. We only add a few comments on them. Table 1 shows that most of the characteristics are common between nos. 5 and 7. However, the boundary of the window is entirely different between the two as is observed in figure 5. In contrast to this case, it is remarkable that the window of no. 6 belongs to a common isomeric class to that of no. 4: $W_4 = \omega W_6$. Since the scaling ratios of the two QLs are related by $\sigma_6 = (\sigma_4)^2$, we obtain $\phi_6 = (\phi_4)^2$ and, consequently, $M_6 = (M_4)^2$ and $\lambda_6 = (\lambda_4)^2$. Note, however, that $\Phi_6 \neq (\Phi_4)^2$.

12. Summary and Discussion

An SSQL with a fractal window is specified by a star. QLs whose windows are t-discs and satisfy the simple-covering condition are considered. If the boundary of the window has a complete division, it is a von Koch curve, whose pieces are generated by an SR, i.e., a strictly self-similar scheme. The relevant window is called a von Koch island. QLs with different windows but with a common SR form an isomeric class. The PIR is a set map acting on the physical space and is specified by the point symmetry, the star and the scaling ratio. The PIR is shown to be a powerful method of producing a part of QLs whose windows are von Koch islands. The conjugate set map to the PIR acts onto the internal space and is a strictly self-similar scheme. It has a unique attractor, which is the window of the relevant QL. A proto-von-Koch-island is associated with the von Koch island; the former develops into the latter by a recursive applications of the conjugate PIR. A heuristic method of obtaining proto-von-Koch-islands is presented. Also, two methods of computer-aided drawing of the relevant QLs are presented. Many windows with fractal boundaries are determined.

Every MLD class of SSQLs includes an infinite number of LI-classes. Each of the LI-classes yields an infinite number of LI-classes in the MLD class if different S-maps and/or star-sigma maps are applied to it. QLs produced by PIRs form surely a minority group among SSQLs. It appears improbable that two QLs produced by PIRs are MLD. Moreover, it is plausible that every MLD class of SSQLs includes one QL which is produced by a PIR. If this is true, we can choose the QL as the representative of the MLD class.

A classification of SSQLs is a big problem, with which the present paper is concerned. The present paper includes a large number of propositions, whose proofs but a few are elementary. The key points for solving the classification problem of proper QLs are: (i) the simple covering condition, (ii) the presence of a CD for the boundary of the window and (iii) the isomerism. The first two of the three are not proven yet, and our classification is restricted to those satisfying them. The author is confident that they are basic properties of every proper QL. He believes that exposition of the three key points above is the major contribution of the present work, which, therefore, can be a breakthrough for a complete classification of SSQLs.

It is easy to prove that every proper window is approximated in any precision by a window determined by a PIR, where the precision is quantified by the Hausdorf distance.

The subgroup C_n of D_n is non-crystallographic, and a star whose point group is C_n yields a QL with this point symmetry; this QL is chiral. A similar comment applies to the case of

D_5, a non-crystallographic and achiral subgroup of D_{10}. It is straightforward to extend the present theory to the three Bravais classes of icosahedral QLs in three dimensions [22].

In the present paper, we confined our argument to the case where the relevant star is proper. QLs of this type form a class of gentle QLs produced by the PIRs. A QL produced by a PIR can have a window with an extremely wild boundary, which is, for example, composed of an infinite number of the connected components. To characterize such a boundary will be a hard task.

A reason mentioned in [10] indicates strongly that almost all the existing models for real quasicrystals will be replaced by models based on the present theory. It is rare, however, that the QL produced by the PIR gives as it is a model for a real quasicrystal. The QL is usually decorated to give a realistic model. A simplest way for the decoration is to apply the S-map. It must be noted that the S-map should be extended in order to produce sites which do not belong to the host module.

Part of the physical-space structures of QLs discussed in this paper are only exhibited because of the limitation of the space. The author has an all-purpose program of drawing the QL produced by the PIR, and the figure of a nominated QL will be provided by the author on request.

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